Announcements

1) Hint on HW 4, problem #3 has been corrected on CTools.

2) HW 4 due on Thursday

Equivalence Relations

(review/aside)

Motivating Example

$$\frac{1}{4} = \frac{5}{20}$$

in the sense that $\frac{5}{20}$ can be reduced to $\frac{1}{4}$, or equivalently,

$$1 \cdot 20 = 4 \cdot 5$$

An equivalence relation on rational numbers is given by

$$\frac{a}{b} = \frac{c}{d} \quad \text{if and only if}$$

$$ad = bc.$$

Definition: (equivalence relation)

Let X be a set. An **equivalence relation** R is a subset of $X \times X$ satisfying

i) $(x,x) \in R \ \forall x \in X$
(reflexivity)

ii) If $(x,y) \in R$, then $(y,x) \in R$
(symmetry)

iii) If $(x,y),(y,z) \in R$, then $(x,z) \in R$
(transitivity)

This usually is written as a "relation" $\sim$ on X satisfying

i) $x \sim x \quad \forall\, x \in X$

(reflexivity)

ii) if $x \sim y$, then $y \sim x$

(symmetry)

iii) if $x \sim y$, $y \sim z$, then $x \sim z$

(transitivity)

Example 1: $\boxed{x \sim y \text{ if } x-y \in \mathbb{Q}}$

(equivalence relation on $\mathbb{R}$)

check!

i) (reflexivity) $x-x = 0 \in \mathbb{Q}$,

so $x \sim x$.

ii) (symmetry) if $x-y \in \mathbb{Q}$,

then $y-x = -1(x-y) \in \mathbb{Q}$

so $y \sim x$ if $x \sim y$.

iii) transitivity

if $x-y \in \mathbb{Q}$, and $y-z \in \mathbb{Q}$,

then

$$x-z = \underbrace{x-y}_{\in \mathbb{Q}} + \underbrace{y-z}_{\in \mathbb{Q}} \in \mathbb{Q}$$

and therefore $x \sim z$ if $x \sim y$ and $y \sim z$. $\square$

Example 2: Consider $\{ f \mid f : \mathbb{R} \to \mathbb{R} \}$

$f \sim g$ if $f(x) = g(x)$ $\forall$ x

outside of a set whose cardinality is at most countable.

For example,

$$f(x) = x^2$$

$$g(x) = \begin{cases} x^2, & x \notin \mathbb{Q} \\ 25, & x \in \mathbb{Q} \end{cases}$$

$f \neq g$ only on $\mathbb{Q}$, which is countable, hence $f \sim g$.

Check:

i) reflexivity $f(x) = f(x) \ \forall x \in \mathbb{R}$, $f \sim f$ (equal off of $\emptyset$).

ii) Symmetry if $f(x) = g(x)$ off of a set $Y \subseteq \mathbb{R}$, then $g(x) = f(x)$ off of Y. Hence $f \sim g$ implies $g \sim f$

iii) transitivity if $f \sim g$, then $\exists$ a subset $A \subseteq \mathbb{R}$, $|A| \leq |\mathbb{N}|$, and $f(x) = g(x)$ on A^c.

If $g \sim h$, then $\exists\ B \subseteq \mathbb{R}$, $|B| \leq |\mathbb{N}|$, $g(x) = h(x)$ $\forall\ x \in B^c$. Then

$|A \cup B| \leq |\mathbb{N}|$, so

on $(A \cup B)^c$,

$f(x) = g(x) = h(x)$, so

$f \sim h$. $\square$

Recall: $\displaystyle\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}, \; |r|<1$

diverges if $|r| \geq 1$.

Note: limit of terms - if $|r|<1$,

then $\displaystyle\lim_{n\to\infty} r^n = 0$.

Proposition: (terms) If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

proof, If $\sum_{n=1}^{\infty} a_n$ converges to L, this means

$$\lim_{k\to\infty} \sum_{n=1}^{k} a_n = \lim_{k\to\infty} S_k = L.$$

$$a_k = \sum_{n=1}^{k} a_n - \sum_{n=1}^{k-1} a_n$$

$$= S_k - S_{k-1}$$

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty}(S_k - S_{k-1})$$

$$= \lim_{k\to\infty} S_k - \lim_{k\to\infty} S_{k-1}$$

(limit exists)

$$= L - L = 0 \quad \square .$$

However, the converse is not true!

Example 3: (harmonic)

$$\sum_{n=1}^{\infty} \frac{1}{n}$$ diverges.

Note $$\lim_{n \to \infty} \frac{1}{n} = 0$$

Why does the series diverge?

Only look at partial sums ending at powers of 2

$$S_1 = 1$$

$$S_2 = 1 + \frac{1}{2}$$

$$S_4 = 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\geq \frac{1}{4} + \frac{1}{4} = 2 \cdot \frac{1}{4} = \frac{1}{2}}$$

$$S_8 = 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\substack{\geq \\ 1/2}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\geq \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = 4\left(\frac{1}{8}\right) = \frac{1}{2}}$$

One can prove, via induction,

that

$$\sum_{n=1}^{2^k} \frac{1}{n} \geq 1 + \frac{k}{2}$$

$$\parallel$$

$$S_{2^k}.$$

Therefore, $\lim_{k \to \infty} S_{2^k} \geq \lim_{k \to \infty} \left(1 + \frac{k}{2}\right)$

$$= \infty$$

So the partial sums have no limit, hence the series diverges.

Example 4: (harmonic squared)

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$ converges!

Again, analyze the partial sums.

Only look at powers of 2.

$$S_1 = 1$$

$$S_2 = 1 + \frac{1}{4}$$

$$S_4 = 1 + \frac{1}{4} + \underbrace{\frac{1}{9} + \frac{1}{16}}_{\overset{?}{\leq}\ \frac{1}{4} + \frac{1}{4} = \frac{1}{2}}$$

$$S_8 = 1 + \frac{1}{4} + \underbrace{\frac{1}{9} + \frac{1}{16}}_{\leq\ 1/2} + \underbrace{\frac{1}{25} + \frac{1}{36} + \frac{1}{49} + \frac{1}{64}}_{\leq\ \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \frac{4}{16} = \frac{1}{4}}$$

$$S_8 \leq 1 + \frac{1}{4} + \frac{1}{2} + \frac{1}{4}$$

Similarly, by induction, we can show

$$S_{2^k} = \sum_{n=1}^{2^k} \frac{1}{n^2}$$

$$\leq 1 + \frac{1}{4} + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{k-1}}$$

$$= 1 + \frac{1}{4} + \sum_{n=1}^{k-1} \left(\frac{1}{2}\right)^n$$

Take limit as $k \to \infty$, we get

$$\lim_{k\to\infty} S_{2^k} \leq 1 + \frac{1}{4} + \lim_{k\to\infty} \sum_{n=1}^{k-1} \left(\frac{1}{2}\right)^n$$

$$= 1 + \frac{1}{4} + 1 = 2 \tfrac{1}{4} \qquad \square$$

Note:
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

Hard and weird!

$$\sum_{n=1}^{\infty} \frac{1}{n^7}$$, no closed form solution is known in terms of "familiar" numbers!

Shift attention to convergence, forget about value of limit